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Nijenhuis tensors and Lie algebras ★

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Abstract

Let M be a $2n$-dimensional almost complex manifold: we construct a local almost complex structure starting from a Nijenhuis tensor given at a point. Moreover, we determine, on a 6-manifold, the conditions ensuring that its Nijenhuis tensor induces a Lie algebra structure on the tangent space. We give a class of examples for every kind.

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0. Introduction

One of the most intriguing problems in complex geometry is to investigate how to deform almost complex structures in order to obtain an integrable one; no obstructions are known for real dimension ≥ 6.

The failure of integrability is fully measured by the Nijenhuis tensor and an interesting subject is to characterize the class of such tensors.

In this paper we study some related questions with special attention to the six-dimensional case. We start by proving that, given a skew-symmetric and J-antibilinear map $\gamma \in \mathrm{Bil}(\mathbb{R}^{2n}, \mathbb{R}^{2n})$, there exists a local almost complex structure on $\mathbb{R}^{2n}$ whose Nijenhuis tensor at the origin is γ. An application of this construction is given in Section 3.1, where γ is the vector product of $\mathbb{C}^3$ (in the canonical identification with $\mathbb{R}^6$). We note that this example

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produces a model of a *totally non-integrable* almost complex structure (see [4]) and the six-dimensional case is the first which may occur.

Since $N_J[x]$ induces an antibilinear skew-symmetric application $T_x^{1,0}M \times T_x^{1,0}M \to T_x^{1,0}M$, it is natural to ask for the condition ensuring a Lie algebra structure on $T_x^{1,0}M$. By the classification of three-dimensional complex Lie algebras (see [6]), in Section 3 we produce a list of typical examples for every kind.

1. Construction of Nijenhuis tensors

Let (M, J) be an almost complex $2n$-manifold and N_J be the Nijenhuis tensor of J,

$$N_J(X, Y) = 2\{[JX, JY] - [X, Y] - J[JX, Y] - J[X, JY]\}.$$

It is immediate to check that the Nijenhuis tensor satisfies the following algebraic conditions:

$$\begin{aligned} N_J(X, Y) &= -N_J(Y, X), \\ N_J(JX, Y) &= N_J(X, JY) = -JN_J(X, Y). \end{aligned} \tag{1.1}$$

We recall the definition of *totally non-integrable almost complex structure* on a manifold M. Let J be an almost complex structure on a $2n$-dimensional manifold M and N_J be its Nijenhuis tensor;

Definition 1.1. An almost complex structure J on a manifold M is said to be *totally non-integrable* if

$$\mathrm{Span}\{N_J[p](X, Y): X,\ Y \in T_pM\} = T_pM.$$

We observe that for $n = 1, 2$ an almost complex structure is never totally non-integrable. In Section 3.1 we will give two examples of such structure.

It is natural to ask: When there exists an almost complex structure J on $\mathbb{R}^{2n}$ such that its Nijenhuis tensor, evaluated at the origin, is equal to a given application $\mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ satisfying (1.1). Consider an almost complex structure J on $\mathbb{R}^{2n}$ and define $\gamma = J_*[0]$, where J_* is the Jacobian of J; γ is a bilinear form on $\mathbb{R}^{2n} \times \mathbb{R}^{2n}$ with values in $\mathbb{R}^{2n}$, i. e. $\gamma \in \mathrm{Bil}(\mathbb{R}^{2n}, \mathbb{R}^{2n})$. Explicitly we have

$$\gamma(X, Y) = J_*[0](X)Y$$

and, for example,

$$\gamma(e_i, e_j) = \begin{pmatrix} \partial_i J_{1j} \\ \vdots \\ \partial_i J_{2n\,j} \end{pmatrix}.$$

By $J^2 = -I$ it follows that

$$\gamma(X, J[0]Y) = -J[0]\gamma(X, Y).$$

Let

$$B_J^{0,2} = \{\gamma \in \mathrm{Bil}(\mathbb{R}^{2n}, \mathbb{R}^{2n}) : \gamma(X, J[0]Y) = -J[0]\gamma(X, Y)\}$$

and define

$$4N_\gamma(X, Y) = \gamma(X, Y) - \gamma(Y, X) - \gamma(J[0]X, J[0]Y) + \gamma(J[0]Y, J[0]X)$$

for $\gamma \in B_J^{0,2}$.

Remark 1.1.
(i) The application $\gamma \overset{N}{\mapsto} N_\gamma$ is an endomorphism of $B_J^{0,2}$;
(ii) $N(N_\gamma) = N_\gamma$.

From this remark it follows that

$$B_J^{0,2} = \mathrm{Ker} N \oplus \mathrm{Im}\, N$$

and

$$\mathrm{Im} N = \{\gamma \in B_J^{0,2} : \gamma(X, Y) = -\gamma(Y, X)\}.$$

Therefore $\mathrm{Im} N$ is the space of bilinear forms satisfying (1.1), i.e. the space of Nijenhuis tensors at a point.

From now on we will assume that $\gamma \in \mathrm{Im}\, N$ is known and we are looking for an almost complex structure J whose Nijenhuis tensor evaluated at the origin is γ.

We may consider J of the form $J[x] = A[x]J_0A^{-1}[x]$ with

$$J_0 = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

and $A[0] = A^{-1}[0] = I$. By a direct computation it follows that

$$4J_0N_{J_*[0]} = -\tfrac{1}{2}N_J[0]$$

and being $N_{J_*[0]} = J_*[0]$, we have

$$J_*[0] = J_0\tfrac{1}{8}N_J[0].$$

We have

$$\gamma(e_i, \cdot) = J_*[0](e_i)\cdot = (A_*[0](e_i)J_0 + J_0(A^{-1})_*(e_i))\cdot = [A_*[0](e_i), J_0] \cdot .$$

If P is a solution of the linear systems

$$\tfrac{1}{8}J_0\gamma(e_i, \cdot) = [P(e_i), J_0] \cdot \quad \text{for } i = 1, \ldots, 2n, \tag{1.2}$$

which we can solve one by one, then we may construct $A[x]$ such that

$$A[0] = I, \qquad A_*[0] = P.$$

Then the almost complex structure $J[x] = A[x]J_0A^{-1}[0]$ satisfies the required condition. Therefore we have proved the following:

Proposition 1.1. *Given any* $\gamma \in \mathrm{Bil}(\mathbb{R}^{2n}, \mathbb{R}^{2n})$ *satisfying* (1.1), *there exists an almost complex structure* J *on* $\mathbb{R}^{2n}$ *such that*

$$J[0] = J_0, \qquad N_J[0] = \gamma.$$

2. Nijenhuis tensors and Lie algebras

One of the most interesting situations is the six-dimensional case; in fact we have the following special construction.

Let (V, J, g) be a six-dimensional real vector space with a complex structure and a J-Hermitian scalar product. With respect to the Hermitian product h_g induced by g on $V^{1,0}$ let $\alpha \in \bigwedge^{3,0} V^*$ with $\| \alpha \|_{h_g} = 1$ and $V_\alpha^g : V \times V \to V$ be defined by

$$\alpha(v, w, u) = h_g(v, V_\alpha^g(w, u)).$$

The map V_α^g is bilinear, satisfies (1.1) and extends to an antisymmetric form on $V^{\mathbb{C}}$ with values in $V^{\mathbb{C}}$ which is $\mathbb{C}$-bilinear with respect to the canonical complex structure and $\mathbb{C}$-biantilinear with respect to the extension of J to $V^{\mathbb{C}}$. Note that, if $\{v_1, v_2, v_3\}$ is a h_g-unitary $\mathbb{C}$-basis with $\alpha(v_1, v_2, v_3) = 1$ then $V_\alpha^g(v_1, v_2) = v_3$, $V_\alpha^g(v_2, v_3) = v_1$ and $V_\alpha^g(v_3, v_1) = v_2$.

We observe that a change of basis preserves this form of V_α^g if and only if the matrix change is in $\mathrm{SO}(3, \mathbb{C})$.

Let $N : V \times V \to V$ be a bilinear map satisfying (1.1). Then there exists $L(N) \in \mathrm{End}(V)$ such that

$$N = L(N) \circ V_\alpha^g, \qquad [L(N), J] = 0$$

(see [4] for more details).

Let (M, J) be a six-dimensional almost complex manifold: in this section we describe the condition in order that a Nijenhuis tensor induces a Lie algebra structure on T_pM.

Let $N : \mathbb{R}^6 \times \mathbb{R}^6 \to \mathbb{R}^6$ be a bilinear map satisfying (1.1), where

$$J = J_0 = \begin{pmatrix} 0 & -I_3 \\ I_3 & 0 \end{pmatrix}$$

and $\{e_1, e_2, e_3, Je_1, Je_2, Je_3\}$ be the canonical basis of $\mathbb{R}^6$ and $k : \mathbb{R}^6 \to \mathbb{C}^3$ the corresponding isomorphism. From now on we will not distinguish between $\mathrm{GL}(3, \mathbb{C})$ and its image in $\mathrm{GL}(6, \mathbb{R})$ (respectively $\mathfrak{gl}(3, \mathbb{C})$ and its image in $\mathfrak{gl}(6, \mathbb{R})$). We define

$$[v, w] = k(N(k^{-1}(\bar{v}), k^{-1}(\bar{w}))) \quad \forall v, w \in \mathbb{C}^3.$$

This map is $\mathbb{C}$-bilinear and skew-symmetric. We have the following:

Lemma 2.1. *Let* $L \in \mathrm{GL}(3, \mathbb{C})$ *and*

$$N(X, Y) = L(V_\alpha^g(X, Y)).$$

Then the previously defined bracket $[\ ,\ ]$ *induces a structure of Lie algebra on* $\mathbb{C}^3$ *if and only if* $L = {}^tL$.

Proof. It is sufficient to check the Jacobi identity on the α-special basis v_1, v_2, v_3. □

Note that the Lie algebra $(\mathbb{C}^3, [\ ,\ ])$ is simple since its derived algebra is equal to the algebra itself and therefore there is no proper ideal.

Remark 2.1. If $L \in \mathrm{End}(\mathbb{C}^3)$ has rank one, then $[\ ,\ ]$ always induces a complex structure of Lie algebra. In fact, in such a case, we may suppose

$$\begin{pmatrix} l_{11} & \lambda l_{11} & \mu l_{11} \\ l_{21} & \lambda l_{21} & \mu l_{21} \\ l_{31} & \lambda l_{31} & \mu l_{31} \end{pmatrix}.$$

Then the Jacobi identity is always satisfied. In particular, by the classification of the three-dimensional complex Lie algebra (see [6]), we have two cases:

(1) The Heisenberg algebra, whose multiplication table is given by

$$[v_1, v_2] = v_3, \qquad [v_2, v_3] = 0, \qquad [v_1, v_3] = 0,$$

which implies that

$$L = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We note that the table is fixed by the change of basis of the following form:

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{11}a_{22} - a_{12}a_{21} \end{pmatrix}.$$

(2) For the other one we may choose a basis $\{v_1, v_2, v_3\}$ such that

$$[v_1, v_2] = v_1, \qquad [v_2, v_3] = 0, \qquad [v_1, v_3] = 0.$$

Therefore

$$L = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and the table is fixed by $A \in \mathrm{GL}(3, \mathbb{C})$,

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 \\ 0 & 1 & 0 \\ 0 & a_{32} & a_{33} \end{pmatrix}.$$

Remark 2.2. Note that the case rank $L = 2$ with $L \in \mathrm{End}(\mathbb{C}^3)$ is related to the complex three-dimensional Lie algebras with bidimensional derived algebra. In such a case we may choose a basis such that

$$[v_1, v_2] = 0, \qquad [v_1, v_3] = \gamma v_1 + \delta v_2, \qquad [v_1, v_3] = \alpha v_1 + \beta v_2$$

with $\alpha\delta - \beta\gamma \neq 0$. Therefore

$$L = \begin{pmatrix} \gamma & -\alpha & 0 \\ \delta & -\beta & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and the table is fixed by the elements $A = (a_{ij}) \in \mathrm{GL}(3, \mathbb{C})$ such that

$$\begin{aligned}
&(a_{21}a_{32} - a_{22}a_{31})\gamma = \alpha(a_{31}a_{12} - a_{11}a_{32}),\\
&(a_{21}a_{32} - a_{22}a_{31})\delta = \beta(a_{31}a_{12} - a_{11}a_{32}),\\
&(a_{21}a_{33} - a_{23}a_{31})\gamma = \alpha(a_{31}a_{13} - a_{11}a_{33}) + (\alpha a_{11} + \beta a_{12}),\\
&(a_{21}a_{33} - a_{23}a_{31})\delta = \beta(a_{31}a_{13} - a_{11}a_{33}) + (\alpha a_{21} + \beta a_{22}),\\
&\alpha a_{31} + \beta a_{32} = 0,\\
&(a_{22}a_{33} - a_{32}a_{13})\gamma = \alpha(a_{32}a_{13} - a_{12}a_{33}) + (\gamma a_{11} + \delta a_{12}),\\
&(a_{22}a_{33} - a_{32}a_{13})\delta = \beta(a_{32}a_{13} - a_{12}a_{33}) + (\gamma a_{21} + \delta a_{22}),\\
&\gamma a_{31} + \delta a_{32} = 0.
\end{aligned}$$

The first two equations imply that

$$(a_{21}a_{32} - a_{22}a_{31}) = (a_{31}a_{12} - a_{11}a_{32}) = 0.$$

Finally if $L \in \mathrm{GL}(3, \mathbb{C})$, then the canonical form of the multiplication table is $[v_i, v_j] = V^g_\alpha(v_i, v_j)$; consequently $L = I$. In such a case the table is fixed by $\mathrm{SO}(3, \mathbb{C})$.

3. Constructions of models

3.1. *Local model for totally non-integrable Nijenhuis tensor*

Consider the map $V^g_\alpha : \mathbb{C}^3 \times \mathbb{C}^3 \to \mathbb{C}^3$ defined in Section 2. With respect to α-special basis $\{e_1, e_2, e_3\}$, we have

$$V^g_\alpha(e_1, e_2) = e_3, \qquad V^g_\alpha(e_3, e_1) = e_2, \qquad V^g_\alpha(e_2, e_3) = e_1.$$

As an application of Proposition 1.1, we start determining an almost complex structure J on a neigbourhood of the origin of $\mathbb{R}^6$, whose Nijenhuis tensor N_J takes the value $N[0] = V^g_\alpha$.

We consider the unknown $J[x]$ of the following form:

$$J[x] = A[x]J_0A^{-1}[x],$$

$A[x]$ being in $\mathrm{GL}(6, \mathbb{R})$ and such that $A[0] = I$. If we impose condition (1.2), then we get a linear system in the unknowns $\partial_h A^{ik}[0]$, whose interesting equations are:

$$\begin{aligned} -\partial_3 A_{21} + \partial_6 A_{51} &= \tfrac{1}{4}, & -\partial_2 A_{31} + \partial_5 A_{61} &= -\tfrac{1}{4}, \\ -\partial_1 A_{32} + \partial_4 A_{62} &= \tfrac{1}{4}, & -\partial_3 A_{12} + \partial_6 A_{42} &= -\tfrac{1}{4}, \\ -\partial_2 A_{13} + \partial_5 A_{43} &= \tfrac{1}{4}, & -\partial_1 A_{23} + \partial_4 A_{53} &= -\tfrac{1}{4}, \end{aligned}$$

$\partial_i A_{jk}$ being evaluated at the origin. Therefore a possible solution of this linear system is given by:

$$\begin{aligned} &\partial_1 A_{23}[0] = \tfrac{1}{8}, && \partial_2 A_{31}[0] = \tfrac{1}{8}, && \partial_3 A_{12}[0] = \tfrac{1}{8}, \\ &\partial_1 A_{32}[0] = -\tfrac{1}{8}, && \partial_2 A_{13}[0] = -\tfrac{1}{8}, && \partial_3 A_{21}[0] = -\tfrac{1}{8}, \\ &\partial_4 A_{62}[0] = \tfrac{1}{8}, && \partial_5 A_{43}[0] = \tfrac{1}{8}, && \partial_6 A_{51}[0] = \tfrac{1}{8}, \\ &\partial_4 A_{53}[0] = -\tfrac{1}{8}, && \partial_5 A_{61}[0] = -\tfrac{1}{8}, && \partial_6 A_{42}[0] = -\tfrac{1}{8}, \end{aligned}$$

and $\partial_i A_{jk}[0] = 0$ otherwise.

Therefore we may choose

$$A[x] = \begin{pmatrix} 1 & x_3/8 & -x_2/8 & 0 & 0 & 0 \\ -x_3/8 & 1 & x_1/8 & 0 & 0 & 0 \\ x_2/8 & -x_1/8 & 1 & 0 & 0 & 0 \\ 0 & -x_6/8 & x_5/8 & 1 & 0 & 0 \\ x_6/8 & 0 & -x_4/8 & 0 & 1 & 0 \\ -x_5/8 & x_4/8 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and consequently $J[x] = A[x] J_0 A^{-1}[x]$ is a local model for a totally non-integrable almost complex structure with $N_J[0] = V_\alpha^g$.

Remark 3.1. Another example of totally non-integrable 6-manifold is the sphere $S^6 = \{x \in \mathcal{I}m\,\mathrm{Cay} \mid \|x\| = 1\}$ endowed with the almost complex structure $J[p]x = px\ \forall p \in S^6$, $\forall x \in T_p S^6$. The Nijenhuis tensor is

$$N_J[p](x, y) = 2((px)y - p(xy)).$$

Therefore we may choose an orthonormal $\mathbb{C}$-basis on $T_p S^6$ such that N_J has the form of V_α^g (see [4] for more details).

3.2. Global model of rank-two Nijenhuis tensor

Let (M, g) be a three-dimensional Riemannian manifold, ∇ be the Levi Civita connection and R its curvature. We recall some facts on the tangent bundle $\pi : TM \to M$ (see [2]). Let

$$T_{(x,u)}TM = H_{(x,u)} \oplus V_{(x,u)}$$

be the splitting induced by ∇ for $(x, u) \in TM$.

Given $X[x] = \sum_{i=1}^{3} X^i(\partial/\partial x^i) \in T_xM$ recall that we may define the *horizontal* and *vertical lift*, respectively, of X as

$$X^h[(x,u)] = \sum_{i=1}^{3}(X^i \circ \pi)\frac{\partial}{\partial \bar{x}^i} - \sum_{i,j,k=1}^{3} \Gamma_{ij}^k X^i u^j \frac{\partial}{\partial u^k} \in H_{(x,u)},$$

$$X^v[(x,u)] = \sum_{i=1}^{3}(X^i \circ \pi)\frac{\partial}{\partial u^i} \in V_{(x,u)},$$

where $(\bar{x}^1, \bar{x}^2, \bar{x}^3, u^1, u^2, u^3)$ is the system of coordinates induced by the local coordinates on M and Γ_{ij}^k are Christoffel's symbols of ∇. Now we may define an almost complex structure J on TM by setting

$$JX^h = X^v, \qquad JX^v = -X^h.$$

We have the following formulas:

$$[X^h, Y^h] = [X,Y]^h - (R_{XY}u)^v, \qquad [X^h, Y^v] = (\nabla_X Y)^v, \qquad [X^v, Y^v] = 0.$$

It follows that

$$N(X^h, Y^h) = (R_{XY}u)^v, \qquad N(X^h, Y^v) = (R_{XY}u)^h, \qquad N(X^v, Y^v) = (R_{XY}u)^v.$$

Remark 3.2. Note that the almost complex structure J on TM may be defined for any dimension of M.

Suppose that M is a three-dimensional space form. Therefore

$$R_{XY}Z = k(g(X,Z)Y - g(Y,Z)X)$$

and $dim_{\mathbb{R}}\mathrm{Span}\{N[(x,u)](X,Y)\colon X,\ Y \in T_{(x,u)}TM\} = 4$. The conditions in order that N induces a Lie algebra on $(T_{(x,u)}TM, J[(x,u)])$ are expressed by the Jacobi identity. Set

$$z_j = \frac{1}{2}\left(\frac{\partial}{\partial x^j}^h - \mathrm{i}J[(x,u)]\frac{\partial}{\partial x^j}^h\right) \quad \text{for } j = 1,2,3.$$

We have

$$[z_j, z_k] = (R_{\partial/\partial x^j \partial/\partial x^k}u)^v + \mathrm{i}(R_{\partial/\partial x^j \partial/\partial x^k}u)^h,$$

that is the bracket defined in the previous section.

By setting $u = \alpha z_1 + \beta z_2 + \gamma z_3$ we have a Lie algebra if and only if

$$\alpha(\gamma z_2 - \beta z_3) = 0.$$

The first case $\alpha = 0$ implies that

$$L = \mathrm{i}\begin{pmatrix} 0 & \gamma & -\beta \\ -\gamma & 0 & 0 \\ \beta & 0 & 0 \end{pmatrix},$$

while the second one $\beta = \gamma = 0$ determines

$$L = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha \\ 0 & -\alpha & 0 \end{pmatrix}.$$

We note that all the three-dimensional complex Lie algebra with two-dimensional derived algebra may be obtained by a suitable choice of α, β, γ.

3.3. Global model of rank-one Nijenhuis tensor

We recall the general construction of the *twistor space* $Z(M, g)$ over an oriented $2n$-dimensional Riemannian manifold (M, g). The six-dimensional twistor spaces will furnish the global model.

Let (M, g) be an oriented $2n$-dimensional Riemannian manifold, $P(M, \mathrm{SO}(2n))$ be the bundle of oriented g-orthonormal frames of M and

$$Z(M, g) = \frac{P(M, \mathrm{SO}(2n))}{\mathrm{U}(n)},$$

with natural projection $\pi : P(M, \mathrm{SO}(2n)) \to Z(M, g)$ be the *twistor space* of M (see [3,5]).

$Z(M, g)$ is a fibre bundle on M with standard fibre $Z(n) = \mathrm{SO}(2n)/\mathrm{U}(n)$ and bundle projection $r : Z(M, g) \to M$.

Let $x \in M$ and $Q \in r^{-1}(x) = Z_x$. Choose $a \in P(M, \mathrm{SO}(2n))/\mathrm{U}(n)$ such that $\pi(a) = Q$. We may define

$$J_x : T_x M \to T_x M, \qquad J_x = a \circ J_0 \circ a^{-1},$$

where J_0 is the canonical complex structure over $\mathbb{R}^{2n}$ and a is viewed as a g-isometry.

The Levi Civita connection on $P(M, \mathrm{SO}(2n))$ induces a splitting

$$T_Q Z(M) = H_Q \oplus V_Q,$$

where

$$V_Q = T_Q Z_{r(Q)} = \{X \in \mathrm{End}(T_{r(Q)} M) \colon X \text{ is } g\text{-antisymmetric } XQ = -QX\}.$$

Defining $\mathcal{J}_Q : T_Q Z_{r(Q)} \to T_Q Z_{r(Q)}$ as

$$\mathcal{J}_Q X = QX,$$

we may introduce an almost complex structure $\mathbb{J}$ on $Z(M, g)$ by setting

$$\mathbb{J}_Q X = (r_*^{-1}|_{r(Q)} \circ J_{r(Q)} \circ r_*|_Q) X^h + QX^v,$$

where $X = X^h + X^v$, with $X^h \in H_Q$, $X^v \in V_Q$.

We recall the following:

Proposition 3.1. *The Nijenhuis tensor $N_{\mathbb{J}}$ of $\mathbb{J}$ is horizontal and vertical valued, i.e.*
(i) $N_{\mathbb{J}}|_Q(X, Y) = 0$ *if X is vertical;*
(ii) $N_{\mathbb{J}}|_Q(X, Y) \in V_Q \; \forall X, Y \in T_Q Z(M)$.

If M is four-dimensional, then its twistor space is six-dimensional. Note that if $Z(M, g)$ is not integrable ($W^+ \neq 0$, see [1]), then the zero locus of Nijenhuis tensor $N_{\mathbb{J}}$ intersects each fibre on four points (see [7]) and then its image is two-dimensional generically. Consequently, $N_{\mathbb{J}}$ induces on $T_Q Z(M, g)$ a complex Lie structure with one-dimensional derived algebra. Proposition 3.1 implies that the derived algebra is contained in the vertical subspace. Therefore, the image of $N_{\mathbb{J}}$ is in the centre of the algebra. We may conclude that the Lie algebra induced is always of the Heisenberg type.

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